# CHAPTER 11

Collider Theory

β Function

The β function is also known as the amplitude function. To understand the concept of the β function we must understand what is occurring as a particle traverses a FODO lattice. We all know that the real world does not follow ideal conditions. Accelerators are no exception. Magnets are not always constructed with perfect field configurations. For example, as magnets are ramped the laminations begin to heat up and the size of the magnet will increase, albeit a small change in volume. The field strength will change or an aberration in the field may become more apparent.

Suppose a proton is injected onto its ideal orbit in a circular accelerator made up of only dipoles and say that one dipole magnet in the ring has a momentary imperfection that causes a deflection in the plane perpendicular to the magnetic field lines. The resulting orbit would be another circle with the same radius but offset from the ideal orbit. The particle will oscillate about the ideal orbit and be considered stable because it remains in the accelerator.
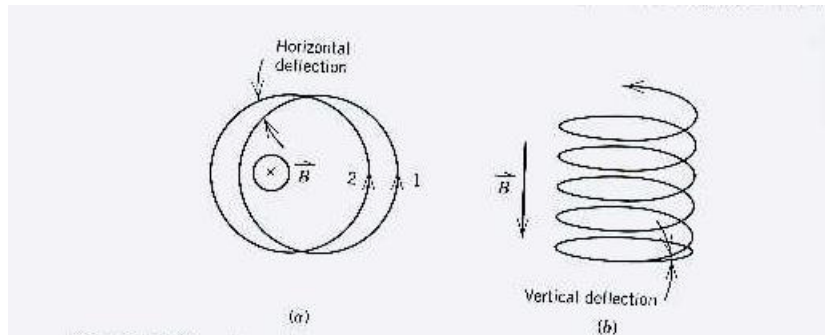
Figure 11.1 The left diagram shows the deflection of a particle in the horizontal plane. The right figure shows the effect if the deflection has components in both transverse planes.

Now consider a deflection from a magnet that has a component parallel to the magnetic field lines. The particles will begin to spiral out of the beam tube. This, of course is an unstable orbit.

The alternating gradient synchrotron was developed to provide strong focusing in both transverse planes. In the Tevatron we accomplish this with the use of quadrupoles arranged in a FODO lattice.
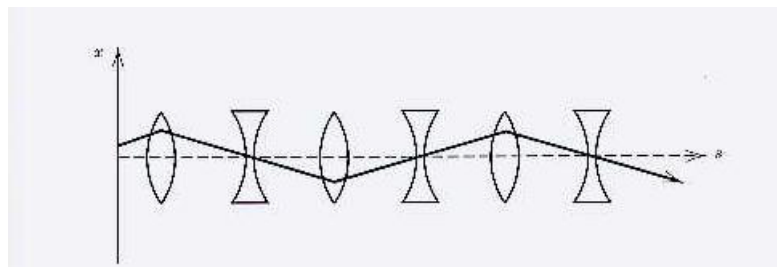
Figure 11.2 Oscillatory motion due to strong focusing by a FODO lattice.

In the above figure the motion of a particle is now periodic due to the placement of quadrupoles amongst the dipoles that keep the particle within the circumference. As with anything that is periodic, it can be compared with the solution for a simple harmonic oscillator.

$$x(s) = A\cos[\Psi(s) + \delta]$$

where A is the amplitude, $\Psi(s)$ is the phase of the particle oscillation, and δ is the phase shift. After some manipulation via matrix applications the general form for the equation of motion of a particle traversing a FODO lattice is

$$x(s) = A\sqrt{\beta(s)}\cos[\Psi(s) + \delta]$$

where $\beta(s)$ is the amplitude function. The amplitude function is interpreted as the local wavelength of the oscillation divided by 2π. This function has units of length and is often quite a large value, on the order of meters, while the actual particle deviation from the ideal orbit is rather small.

The number of oscillations the β function goes through in one revolution of the accelerator is called the tune, ν.

Gaussian Distribution and Luminosity

In particle physics a colliding beams experiment has a great advantage over a fixed-target experiment due to the center-of-mass energy attainable for the creation of new particles.

Diagram of FT and colliding beam CMS

In fixed-target the center-of-mass energy available for new secondary particle creation goes as the square root of the initial proton's energy, $E^{\frac{1}{2}}$. As the proton's energy is increased the gain in the secondary particles energy is small. However, in the colliding beam center-of-mass frame the proton and antiproton annihilate upon collision and give their total energy to the creation of new particles. So for a 980 GeV beam of protons and antiprotons the total available energy for new secondaries is 1.96 TeV.

There are disadvantages, too, in colliding two beams. The particles must be stable, although a muon collider has been considered due to the "long" lifetime of that particle. In the fixed-target experiments the proton collision rate with the target is high, whereas in the colliding beam experiment the collision rate is low.
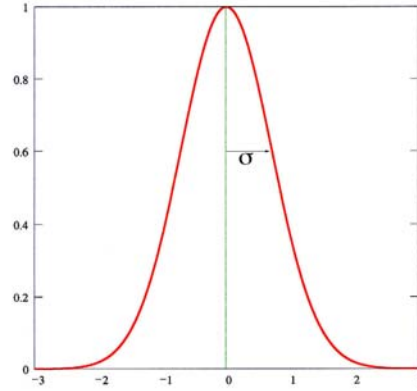
Figure 11.3 Gaussian shape of the beam. The majority of protons/pbars reside at the correct energy and position. Some particles have higher/lower momentum and position. The vertical axis is the number of particles

The two beams as they are counter rotating in the accelerator ring have a Gaussian shape, hopefully. Refer to the picture above. Each particle has a probability of interacting with another particle traveling in the opposite direction. This is known as the interaction cross-section, σ_{int}. The rate of interaction within a detector is given by

$$R = \sigma_{int} L$$

where L is the luminosity. So what is luminosity? Lets define that now. The luminosity is a measure of how the particles in both bunches are interacting with each other. It is dependent upon the revolution frequency and the area that the beam occupies.

$$L = \frac{fnN_p N_{\bar{p}}}{A}$$

where N_p and N_{pbar} are the number of particles in each bunch, f is the revolution frequency, n is the number of bunches in either beam, and A is the cross-sectional area of the beams. Since the antiproton bunches and proton bunches can have different cross-sectional areas, A can be defined in terms of the width of the Gaussian shape, σ_p and σ_{pbar}. The luminosity in the Tevatron is defined as

$$L = \frac{fnN_p N_{\bar{p}}}{2\pi\left(\sigma_p^2 + \sigma_{\bar{p}}^2\right)} F\left(\frac{\sigma_l}{\beta^*}\right)$$

where f, n, N_p, and N_{pbar} are the same as defined above. The denominator contains σ_p and σ_{pbar}, which is the standard deviation of the beam spacially at the interaction point in the detector. This is just a measure of the width for the bunch. $F(\sigma_l/\beta^*)$ is a form factor (a

percentage) dependent upon the bunch length, σ_l, and the beta function at the interaction point, β^*. Referring to the table below, in Run II β^* is 35 cm and σ_l is 0.37m.

	RUN IB	RUN II with MI	RUN II with MI + Recycler	
Protons/bunch	2.32E+11	2.70E+11	2.70E+11	
Antiprotons/bunch	5.50E+10	3.00E+10	7.00E+10	
Total antiprotons	3.30E+11	1.30E+12	2.50E+12	
Pbar production rate	6.00E+10	1.70E+11	2.00E+11	pbars/hr
Proton emittance	23π	20π	20π	mm-mr
Antiproton emittance	13π	15π	15π	mm-mr
β*	0.35	0.35	0.35	mm-mr
Energy	900	1000	1000	GeV
Bunches	6	36	36	
Bunch length (rms)	0.6	0.43	0.38	mm-mr
Form Factor	0.59	0.7	0.7	
Typical Luminosity	1.60E+31	8.10E+31	2.00E+32	$cm^{-2}sec^{-1}$
Integrated Luminosity	3.2	16.3	41	pb^{-1}/week
Bunch spacing	3500	396	396	nsec
Interactions/crossing	2.7	2.3	5.8	at 50 mb
Pbar tune shift	0.015	0.02	0.02	Horizontally
Proton tune shift	0.006	0.003	0.007	Horizontally

A high luminosity is what Operations strives for because it will yield a large interaction rate. By looking at the Luminosity equation above it can be seen that the luminosity increases as the intensity per bunch increases. Also, if the bunch cross-sectional area decreases then the luminosity increases. The average luminosity in Run Ib was $1.6x10^{31}$ $cm^{-2}sec^{-1}$. The luminosity goal for Run II is $5x10^{32}$ $cm^{-2}sec^{-1}$. This will be achieved due to the larger N_p from the Main Injector and an increased N_{pbar} from the Recycler.

The performance of a collider is determined by the integrating the luminosity over time. This yields units of cross section, which are units of barns (1 b = 10^{-24} cm^2).

Emittance

We are at a point where the topic of emittance can be discussed. As stated in the β function section, the solution to periodic motion through a FODO lattice is

$$x(s) = A\sqrt{\beta(s)}\cos[\Psi(s)+\delta]$$

11.4

By taking the derivative of the above equation, x'(s), and plotting its value against x(s) we obtain a phase space diagram.
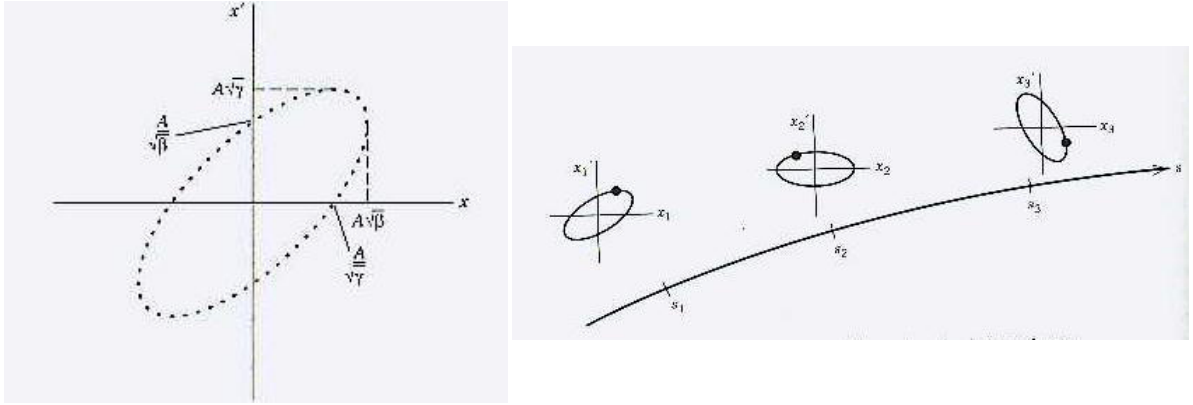

Figure 11.4 The left figure shows the phase space diagram of a stable orbit. The right side figure shows the rotation of the ellipse throughout the accelerator. Notice the area of the ellipse is unchanged.

At any point in the accelerator, the maximum value of x is $A\beta^{1/2}$. The area of the phase space remains the same but the ellipse rotates with respect to the position in the ring. The phase space occupied by the beam is called the emittance, ε. If all the particles were on the ideal orbit then the emittance would be zero because all of the particles would reside at one point on the phase space diagram. If the particles in the beam have a Gaussian distribution then the emittance is

$$\varepsilon = -2\pi\sigma^2 \frac{\ln\left(1 - 6\pi\sigma^2/\beta\right)}{\beta}$$

where σ is the width of the Gaussian defined earlier in this chapter. The above equation gives the phase space that contains 95% of the beam. The units associated with emittance are mm-mr (mr = milliradians).
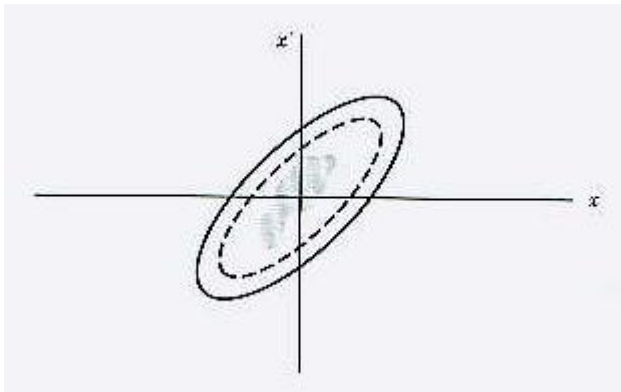
Figure 11.5 Phase space and its relation to the beam within.

RF Theory

The previous section dealt with the motion in the transverse plane. Now the equations of motion in the longitudinal direction will be developed.

The progress of a particle through an accelerator can be charted via a phase space diagram of the longitudinal direction (z-axis). Let τ be the time of flight of the ideal particle passing through an RF station in one turn.

$$\tau = \frac{C}{v}$$

where C is the circumference and v is the velocity of the particle. The fractional change in τ is then

$$\frac{\Delta\tau}{\tau} = \frac{\Delta C}{C} - \frac{\Delta v}{v}$$

In relativistic terms

$$\frac{\Delta v}{v} = \frac{1}{\gamma^2}\frac{\Delta p}{p}$$

where p is the momentum and γ is $\dfrac{1}{\sqrt{1-\left(v/c\right)^2}}$. The first term in the fractional time equation also depends on the momentum deviation. Of course, more than one particle is accelerated and statistically some will be slightly higher and lower in momentum which implies there will be various orbits about the ideal orbit. A new parameter, γ_t, is introduced.

$$\frac{\Delta C}{C} = \frac{1}{\gamma_t^2}\left(\frac{\Delta p}{p}\right)$$

The value of γ_t is actually determined in the design of an accelerator. For the Tevatron, γ_t is 18. Thus the expression of the fractional change in τ is

$$\frac{\Delta\tau}{\tau} = \frac{1}{\gamma_t^2}\left(\frac{\Delta p}{p}\right) - \frac{1}{\gamma^2}\left(\frac{\Delta p}{p}\right)$$

The term $\dfrac{\Delta\tau}{\tau}$ is called the slip factor, η.

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

Now you can see that when $\gamma=\gamma_t$ the sign of η transitions towards a positive number. This occurs at the transition energy. Luckily, for the TeV the beam injected is already above the transition energy. The longitudinal equations of motion can now be constructed.

Suppose a particle arrives at the nth accelerating station with the energy and phase E_n and ϕ_n. At the entrance to the $(n+1)^{th}$ cavity the energy and phase are E_{n+1} and ϕ_{n+1}.

$$\frac{\Delta\tau}{\tau} = \eta\frac{\Delta p}{p}$$

The angular RF frequency, ω_{rf}, multiplied by the time of flight yields $2\pi h$, where h is the harmonic number of the TeV, 1113.

$$\frac{\Delta\phi}{2\pi h} = \eta\frac{\Delta p}{p} = \frac{\eta}{\beta^2}\frac{\Delta E}{E}$$

$$\phi_{n+1} = \phi_n + \frac{2\pi h\eta}{\beta^2}\frac{\Delta E}{E}$$

The above phase equation is one of our equations of motion in the longitudinal direction. The next equation deals with the energy of the ideal particle. Every time the particle traverses the RF cavity it gains energy,

$$\left(E_s\right)_{n+1} = \left(E_s\right)_n + eV\sin\phi_s$$

where e is the charge of an electron, V is the amplitude of the emf across the cavity's gap, and ϕ_s is the phase for arrival of the ideal particle, aka the synchronous phase. For any particle on any orbit the energy gain as it traverses the cavity is

$$E_{n+1} = E_n + eV\sin\phi_n$$

and so the change in energy between any particle and the ideal particle is

$$\Delta E_{n+1} = \Delta E + eV\left(\sin\phi_n - \sin\phi_s\right).$$

This is the second equation of motion. These equations transcribe orbits on a phase plot, ΔE vs. ϕ, which show where particles in a beam are on stable and unstable orbits.
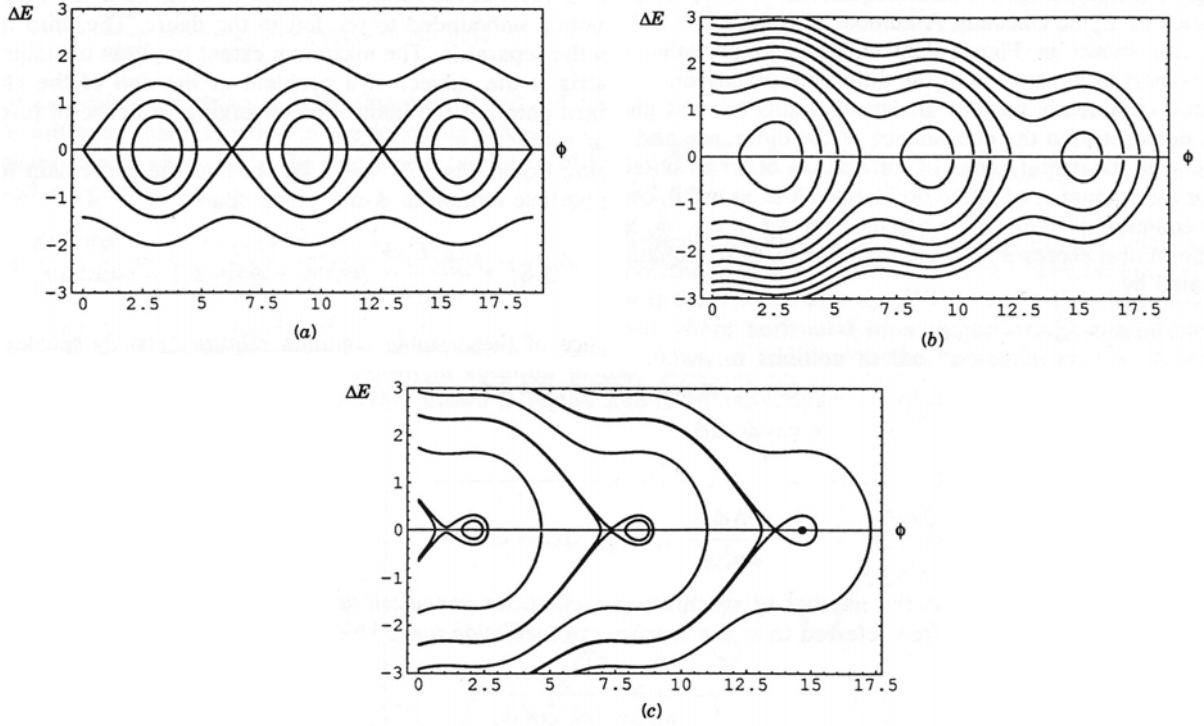
Figure 11.6 Longitudinal phase space development. Particles are injected into a stationary bucket (a) and as the beam is accelerated (b) the phase space of the bucket shrinks until finally (c) the beam reaches the destination energy of 980 GeV.

The above equations form a second order differential equation.

$$\frac{d^2\Delta\phi}{dn^2} - \frac{2\pi h\eta eV\cos\phi_s}{\beta^2 E_s}\Delta\phi = 0$$

From this equation the synchrotron frequency is found to be

$$\Omega_s = \sqrt{\frac{-h\eta eVc\cos\phi_s}{2\pi C^2 E_s}}$$

where c is the speed of light in a vacuum, C is the circumference of the TeV, and the other parameters have been previously defined. Notice from the equation that as the energy increases the synchrotron oscillations decrease. If we plug in the values for the TeV then we find

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$$\Omega_s \approx 100 Hz$$

$$;h = 1113$$

$$\eta = \left(\frac{1}{18^2}\right)$$

$$c = 3 \times 10^8 \, m/s$$

$$\phi_s = \pi$$

$$C = 2\pi \times 10^3 \, m$$

$$E_s = 150 \times 10^9 \, eV$$

ϕ_s is π because the TeV is above transition.